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Instability of a liquid film moving under the effect of gravity and gas flow

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Abstract—The model two-wave equation for weakly nonlinear long waves on a liquid film moving along an inclined plane under the effect of gravity and gas flow is derived on the basis of an integral approach. The linear stability of a liquid film flowing concurrently or countercurrently with a gas stream is studied over a wide range of regime parameters. It is found that the dispersive curve has two branches which interact with one another under certain conditions.

1. INTRODUCTION

The combined motion of a laminar liquid film and turbulent gas is considered as one of the fundamental regimes of two-phase flows. The peculiarity of gas–film flows is the interface instability which results in the formation of nonlinear waves that essentially influence the drag and heat and mass transfer. Until now the problem of describing the wave motion of the liquid film in the presence of a gas flow is far from complete in spite of a large number of studies [1–5]. The basic reasons for this are as follows: a wide variety of wavy regimes; a predominantly nonlinear character of waves; the complexity of setting the boundary conditions on the curvilinear interface allowing for turbulence in a gas phase.

Most theoretical studies are devoted to the linear analysis of instability on the basis of the Orr–Sommerfeld equation [6–9]. The major attention is given to the problem of determining the stresses which are caused by a turbulent gas flow on a liquid film surface. The possibility for the study of nonlinear wave formation has arisen rather recently owing to the development of the numerical methods [10, 11].

For such complicated systems as liquid films with gas flows the model wave equations are of great importance. In spite of limitations, they have significant advantages connected with the relative simplicity and convenience of mathematical treatment. The most complete analysis of the model equations for a falling film is presented in the monograph by Alekseenko *et al.* [4].

In this work the model two-wave equation for weakly nonlinear long waves on a liquid film moving along an inclined plane under the effect of gravity and the gas flow is derived on the basis of the integral approach. The equation is valid over a wide range of film parameters. In the limiting and particular cases it transforms to the known wave equations. The linear analysis of film stability is carried out for both vertical and horizontal gas–liquid flow.

2. UNPERTURBED SOLUTIONS FOR THE MOMENTUM TRANSFER EQUATIONS

First we consider a steady-state flow of a smooth laminar liquid film in the presence of a gas flow. The effect of the gas is taken into account through the given values of shear τ_s and normal p_s stress on a free surface. The Navier–Stokes equations and boundary conditions are written as follows:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}+g\cdot\sin\theta=0 \qquad (1)$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}-g\cdot\cos\theta=0 \qquad (2)$$

$$u=0 \quad \text{when} \quad y=0$$

$$p=p_s(x); \quad \mu\frac{\partial u}{\partial y}=\tau_s \quad \text{when} \quad y=h.$$

Here ρ is the liquid density, ν, μ are the kinematic and dynamic coefficients of viscosity, g is the free fall acceleration, h is the film thickness, u is the longitudinal velocity, p is the pressure, x and y are the Cartesian coordinates, θ is the angle of film flow inclination. The most general case is considered when the velocity vectors of liquid and gas are both in positive and negative directions of the x-axis in an arbitrary combination.

From the second equation we have the hydrostatic distribution of pressure across the film:

$$p_0=p_{s0}(x)+\rho g\cdot(h_0-y)\cdot\cos\theta.$$

With allowance for this expression and the boundary conditions, the parabolic velocity profile follows from equation (1):

NOMENCLATURE

$A\,B\,D\,F$ coefficients, equation (23)
c phase velocity
c_0 velocity of kinematic wave
$c_1\,c_2$ velocities of dynamical waves
f $= u/U$, dimensionless velocity
Fi $= \sigma^3/\rho^3 g\nu^4$, film number
Fr $= g|\cos\theta|h_0^3/q_0^2$, Froude number
g acceleration of gravity [m s^{-2}]
h film thickness [m]
H perturbation of the film thickness
j coefficient
k $= 2\pi h_0/\lambda$, wave number
$l\,L$ scales of length [m]
m coefficient
n_0–n_9 coefficients, equation (34)
p pressure in the liquid [Pa]
p_s gas pressure at the interface
$\hat{p}_{sR}\,\hat{p}_{sI}$ amplitudes of pressure perturbations
q flow rate of the liquid per unit width of a film [m^2 s^{-1}]
Q perturbation of the liquid flow rate
r parameter
Re $= |q_0|/\nu$, Reynolds number
t time [s or dimensionless]
T shape factor
TM $= (\tau_{s0}/\rho)\cdot(3/g\nu)^{2/3}$, gas stream parameter
u liquid velocity [m s^{-1}]
u_a average velocity of the liquid [m s^{-1}]
U liquid velocity at the interface
We $= \sigma h_0/\rho q_0^2$, Weber number
x longitudinal coordinate [m or dimensionless]
y transverse coordinate [m].

Greek symbols
α shape factor, equation (3)
β increment (growth rate factor) shape factor, equation (4)
ε $= h/L$, long-wave parameter
η $= y/h$, dimensionless coordinate
θ inclination angle of the film flow [rad]
κ perturbation of the shape factor χ
λ wavelength [m]
μ viscosity [kg m^{-1} s^{-1}]
ν kinematic viscosity [m^2 s^{-1}]
ρ liquid density [kg m^{-3}]
σ surface tension [kg m s^{-1}]
τ shear stress [N m^{-2}]
τ_s shear stress at the interface
$\hat{\tau}_{sR}\,\hat{\tau}_{sI}$ amplitudes of the shear stress perturbations
χ shape factor, equation (5).

Subscripts
s interface
w wall
0 unperturbed value.

$$u_0 = \frac{(\tau_{s0}+\rho g h_0 \sin\theta - h_0\cdot \mathrm{d}p_{s0}/\mathrm{d}x)}{\mu}y - \frac{(\rho g \sin\theta - \mathrm{d}p_{s0}/\mathrm{d}x)}{2\mu}y^2. \quad (3)$$

Hereafter unperturbed quantities will be denoted by subscript '0'. We shall also write out some useful relations:

$$q_0 \equiv \int_0^{h_0} u_0\,\mathrm{d}y = \frac{\tau_{s0}h_0^2}{2\mu} + \frac{(\rho g \sin\theta - \mathrm{d}p_{s0}/\mathrm{d}x)}{3\mu}h_0^3 \quad (4)$$

$$U_0 = u_0|_{y=h_0} = \frac{\tau_{s0}h_0}{\mu} + \frac{(\rho g\sin\theta - \mathrm{d}p_{s0}/\mathrm{d}x)}{2\mu}h_0^2 \quad (5)$$

$$\tau_{w0} \equiv \mu\frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{3\mu q_0}{h_0^2} - \frac{\tau_{s0}}{2} \quad (6)$$

where q_0 is the flow rate per unit of film width, U_0 is the surface velocity, τ_{w0} is the wall shear stress.

It is convenient to represent velocity profile (3) in dimensionless form excluding gravity and the pressure gradient using relation (5):

$$\frac{u_0}{U_0} = (2-T_0)\frac{y}{h_0} + (T_0-1)\frac{y^2}{h_0^2}. \quad (7)$$

Here

$$T_0 = \tau_{s0}h_0/\mu U_0. \quad (8)$$

In contrast to the falling liquid film, the allowance for stress on a free surface leads to a non-self-similar velocity profile. The quantity T_0 plays the role of the shape factor. Expression (8) can be written alternatively using relations (4)–(6) or the relation between shear and normal stresses which follows from the solution of the problem for the gas phase.

The obtained solutions may be simplified if one takes into account that in practice, the combined flow of liquid and gas is usually realized in a channel of height $l \gg h$. Then the force balance $2\tau_s \approx -l\,\mathrm{d}p_s/\mathrm{d}x$ gives the condition:

$$\frac{|\tau_s|}{h} \gg \left|\frac{\mathrm{d}p_s}{\mathrm{d}x}\right|. \quad (9)$$

With allowance for equation (9), equation (4) takes the form:

$$q_0 = \frac{\tau_{s0}h_0^2}{2\mu} + \frac{\rho g \sin\theta}{3\mu}h_0^3. \quad (10)$$

This equation serves to determine the unperturbed thickness h_0 from the given values of flow rate q_0 and shear stress τ_{s0}.

3. INTEGRAL EQUATIONS

We consider nonlinear long waves on the interface assuming the wavelength $\lambda \gg h$. As is shown [2, 4, 12, 13], the application of the integral correlation method is justified for a film at moderate Reynolds numbers and in the long-wave approximation. This method consists of writing the boundary layer equations with allowance for a free surface and integrating them across the film thickness. To solve the integral equations, it is necessary to use certain assumptions for the instantaneous velocity profile. So, in the case of weakly nonlinear quasi-stationary waves on a free falling liquid film, the velocity profile is approximated by the self-similar polynomial of degree 2. For gas–liquid flow significant difficulties arise connected with the allowance for the non-self-similarity of the instantaneous velocity profile.

In the case of the usually accepted statement [2, 3], the gas and liquid flows may be considered separately. Then the main problem is to determine boundary conditions at the perturbed interface. Here we are interested mainly in the liquid flow. Because of this, the boundary conditions at the interface are given on the basis of known theories (quasi-laminar [10, 11] and relaxation [14] models).

We write out immediately the two-dimensional integral equations for the problem under consideration using the results [4, 13] for a falling film:

$$\frac{\partial}{\partial t}\int_0^h u\,\mathrm{d}y+\frac{\partial}{\partial x}\int_0^h u^2\,\mathrm{d}y=\frac{\tau_s-\tau_w}{\rho}$$
$$-\frac{h}{\rho}\frac{\partial p_s}{\partial x}-gh\frac{\partial h}{\partial x}\cos\theta+gh\cos\theta+\frac{\sigma h}{\rho}\frac{\partial^3 h}{\partial x^3} \quad (11)$$

$$\frac{\partial h}{\partial x}+\frac{\partial}{\partial x}\int_0^h u\,\mathrm{d}y=0. \quad (12)$$

The only difference from a falling film is that the additional terms, i.e. shear stress τ_s and pressure gradient, appear in the right-hand side of equation (11). Here τ_s, τ_w, p_s, h as well as local flow rate $q=\int_0^h u\,\mathrm{d}y$, averaged over thickness velocity $u_a=q/h$ and surface velocity U, are functions of coordinate x and time t.

So far we have not yet determined instantaneous velocity distribution, but we give it by nondimensional coefficients, putting

$$f=u/U,\quad \eta=y/h.$$

Then we have the following relations:

$$\alpha\equiv\int_0^1 f\,\mathrm{d}\eta=q/hU=u_a/U \quad (13)$$

$$\beta\equiv\int_0^1 f^2\,\mathrm{d}\eta \quad (14)$$

$$\chi\equiv\frac{1}{hu_a^2}\int_0^h u^2\,\mathrm{d}y=\frac{\beta}{\alpha^2}. \quad (15)$$

Assume that the dimensionless local velocity distribution is described by expression (7), which takes the form in accepted designations

$$f=(2-T)\eta+(T-1)\eta^2. \quad (16)$$

This distribution is not self-similar, therefore the quantities α, β, χ will depend on $T=T(x,t)$.

Hereafter we shall use dimensionless quantities. For this, the following scales are introduced: $|q_0|$, h_0, $u_{a0}=|q_0|/h_0$, where index '0' denotes the unperturbed values. Let us use dimensionless quantities as follows:

$$x/h_0\to x\quad y/h_0\to y\quad tu_{a0}/h_0\to t$$
$$h/h_0\to h\quad u/u_{a0}\to u$$
$$p/\rho u_{a0}^2\to p\quad \tau/\rho u_{a0}^2\to\tau\quad q/|q_0|\to q.$$

We introduce also the nondimensional numbers:

Reynolds number $Re=|q_0|/\nu$,

Froude number $Fr=g|\cos\theta|h_0/u_{a0}^2$,

Weber number $We=\sigma/\rho h_0^2u_{a0}^2$,

Instead of the Weber number, the film number $Fi=\sigma^3/\rho^3g\nu^4$ is often used. As is well known, the boundary layer approach is valid at $Re\sim 1/\varepsilon\gg 1$, where $\varepsilon=h_0/L\ll 1$, length scale $L\sim\lambda$. However it is noted in refs. [4, 13] that in the case of long-wave processes in a liquid film, equations (11) and (12) may be applied for a wider range of Reynolds numbers: $1<Re<1/\varepsilon^2$.

Then the integral equations are written in non-dimensional form as

$$\frac{\partial q}{\partial t}+\frac{2\chi}{h}q\frac{\partial q}{\partial x}+\frac{q^2}{h}\frac{\partial\chi}{\partial x}$$
$$-\frac{q^2}{h^2}\chi\frac{\partial h}{\partial x}=\tau_s-\tau_w-h\frac{\partial p_s}{\partial x}$$
$$-m\cdot Fr\cdot h\frac{\partial h}{\partial x}+mFr\,h\tan\theta+We\cdot h\frac{\partial^3 h}{\partial x^3} \quad (17)$$

$$\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=0. \quad (18)$$

Here $m=\cos\theta/|\cos\theta|$.

Information on the velocity distribution is contained in the quantity $\chi=\chi(x,t)$, which may be considered as a shape factor and which is designated by Γ in [2]. The shear stress τ_w is also determined by the instantaneous velocity profile. However in the quasi-equilibrium approximation we exclude τ_w similar to ref. [15] by application of equilibrium relation (6) written in dimensionless form:

$$\tau_w=3q/h^2Re-\tau_s/2. \quad (19)$$

The system of nonlinear nonstationary equations for local values of flow rate q and thickness h is solvable when free surface stresses $\tau_s(x,t)$ and $p_s(x,t)$ are given and the velocity profile in a film (or shape factor $\chi(x,t)$) is also determined.

4. TWO-WAVE EQUATION

We consider the weakly nonlinear waves assuming

$$h = 1+H \quad q = j+Q \quad \chi = \chi_0+\kappa \tag{20}$$

where perturbations $|H|$, $|Q| \ll 1$, $|\kappa| \ll \chi_0$, $j = q_0/|q_0|$. If the flow rate is given as that in the experiment, then the average value of flow rate perturbation $\langle Q \rangle = 0$, whereas $\langle H \rangle \neq 0$ and $\langle \kappa \rangle \neq 0$ owing to the nonlinearity of the process.

The interface stresses may be represented according to [2] as follows:

$$\tau_s = \tau_{s0}+\tau_s' = \tau_{s0}+\hat{\tau}_{sR}H+\hat{\tau}_{sI}H_x \tag{21}$$

$$p_s = p_{s0}+p_s' = p_{s0}+\hat{p}_{sR}H+\hat{p}_{sI}H_x. \tag{22}$$

Here $H_x \equiv \partial H/\partial x$ and $\hat{\tau}_{sR}$, $\hat{\tau}_{sI}$, $\hat{p}_{sR}$, $\hat{p}_{sI}$ are the amplitudes of shear stress and pressure perturbations at the interface. Such representation is possible in the case of a linear response of the gas phase to an interface perturbation that is the commonly accepted assumption [2, 10, 11]. In the given case the coefficients in equations (21) and (22) will be considered to be known. The methods of their calculation are given, for example, in refs. [10, 11, 14]. It is extremely important to note that the effect of a gas on the film stability appears mainly owing to a change of stresses along the perturbed interface. Expansions (21) and (22) are convenient to use for the derivation of model equations as well as for the interpretation of film instabilities caused by gas flows. Note that the expansions of τ_s and p_s are written out in equations (21)–(22) with respect to the unperturbed values. Hanratty [2] has performed such a procedure with respect to the mean values. These differences are of no significance for weakly nonlinear waves.

We multiply equation (17) by h^2. Then leaving the quadratic nonlinearity alone and taking into account equation (19) we get

$$\begin{aligned} &Q_t+2j\chi_0Q_x+\kappa_x-AH_x-BH+3Q/Re \\ &+\hat{p}_{sI}H_{xx}-WeH_{xxx} = -2HQ_t-2j\kappa Q_x \\ &-2\chi_0QQ_x-2j\chi_0HQ_x-2jQ\kappa_x-H\kappa_x+\kappa H_x \\ &+2jQH_x\chi_0+DH^2+FHH_x-3\hat{p}_{sI}HH_x+3WeHH_{xxx} \end{aligned} \tag{23}$$

where

$$A = \chi_0+3\hat{\tau}_{sI}/2-\hat{p}_{sR}-m\cdot Fr$$

$$B = \tfrac{3}{2}\hat{\tau}_{sR}+\frac{9j}{Re}-\tfrac{3}{2}\tau_{s0}$$

$$D = 3(\hat{\tau}_{sR}+\frac{3j}{Re}-\tau_{s0})$$

$$F = 3(\hat{\tau}_{sI}-\hat{p}_{sR}-m\cdot Fr).$$

Here it is taken into account that for unperturbed values

$$\tfrac{3}{2}\tau_{s0}-\frac{3j}{Re}+mFr\cdot\tan\theta-\frac{dp_{s0}}{dx} = 0. \tag{24}$$

From equation (18) it follows

$$h_t+Q_x = 0. \tag{25}$$

We shall reduce system (23)–(25) to one equation for H. For this, we differentiate equation (23) with respect to x and eliminate Q_x using equation (25). The quantity Q in nonlinear terms is eliminated by application of the approximated (quasi-stationary) relation

$$Q = cH, \tag{26}$$

where c is the phase velocity which is a weakly changing function for quasi-stationary processes.

Expression (26) follows from equation (25) under the condition

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial x} \tag{27}$$

and it is exact for stationary waves. It turns out that the phase velocity c arises in the right-hand side of equation (23) in combination with $\partial/\partial x$ only. Therefore the operator $c\,\partial/\partial x$ is replaced inversely by $(-\partial/\partial t)$ according to equation (27). This approximate procedure is fulfilled only for the nonlinear terms which have a higher infinitesimal order compared to the linear terms.

As a result one equation is obtained

$$\begin{aligned} &-[H_{tt}+2j\chi_0H_{tx}+AH_{xx}+3H_t/Re \\ &+BH_x-\hat{p}_{sI}H_{xxx}-\kappa_{xx}+WeH_{xxxx}] = 2(1-\chi_0)(HH_t)_t \\ &+2j(\kappa H)_{tx}+(\kappa H_x)_x-(H\kappa_x)_x+2HH_xD \\ &+F(HH_x)_x-3\hat{p}_{sI}(HH_{xx})_x+3We(HH_{xxx})_x. \end{aligned} \tag{28}$$

However it contains two functions—perturbations of thickness H and shape factor κ. The next step is to express the quantity κ through H. To do this, it is necessary to make an assumption for velocity distribution in a liquid film. In the approximation of local quasi-equilibrium, the instantaneous velocity profile is described by parabolic polynomial equation (16) which is nonself-similar owing to the allowance for interface stress. Then the expression for the local value of shape factor χ follows from equations (13)–(15):

$$\chi = \frac{6}{5}\left[1+\frac{T}{(T-4)^2}\right] \tag{29}$$

where

$$T = 4/(1+6q/Re\tau_sh^2). \tag{30}$$

The last formula is obtained from functional relations (8), (4) and (5) rewritten in dimensionless form for instantaneous values. Thus the index zero is omitted.

Now we assume

$$\chi = \chi_0+\kappa \quad T = T_0+T'. \tag{31}$$

Here κ and T' are the perturbations of shape factors.

Unperturbed quantities with allowance for equation (20) have the form:

$$\chi_0 = 6[1+T_0/(T_0-4)^2]/5 \quad (32)$$

$$T_0 = 4/(1+6/jRe\cdot\tau_{s0}). \quad (33)$$

Using rather cumbersome transformations we find from equations (29)–(33), (20) and (21) in the approximation of quadratic nonlinearity the functional relation

$$\kappa = \kappa(Q, H, H_x).$$

Substituting this expression into equation (28) and leaving only quadratic terms we eliminate Q the same way as in the deduction of equation (28). As a result we obtain one equation for the perturbation of film thickness H:

$$\frac{3}{Re}\left(\frac{\partial}{\partial t}+c_0\frac{\partial}{\partial x}\right)H+\left(\frac{\partial}{\partial t}+c_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+c_2\frac{\partial}{\partial x}\right)H$$
$$+n_0H_{xxx}+WeH_{xxxx} = n_1HH_x+n_2(HH_x)_x$$
$$+n_3(HH_t)_t+n_4(HH_t)_x+n_5H_xH_{xx}+n_6HH_{xxx}$$
$$+n_7(HH_x)_{tx}+n_8(H_xH_{xx})_x+n_9(HH_{xxx})_x \quad (34)$$

which describes two-dimensional long weakly nonlinear nonstationary waves on the surface of a liquid film moving along an incline plane under the effect of gravity and gas flow. Here

$$c_0 = 3j+Re(\hat{\tau}_{sR}-\tau_{s0})/2 \quad (35)$$

$$c_{1,2} = (1.2j+Re\cdot\tau_{s0}/40)\pm\{(1.2j+Re\cdot\tau_{s0}/40)^2$$
$$-1.2+mFr+\hat{p}_{sR}-3\hat{\tau}_{sI}/2+Re\cdot\tau_{s0}(2Re\cdot\hat{\tau}_{sR}$$
$$+3Re\cdot\tau_{s0}+6j+6\hat{\tau}_{sR}j/\tau_{s0})/120\}^{1/2} \quad (36)$$

$$n_0 = -\hat{p}_{sI}-Re\cdot\hat{\tau}_{sI}(Re\cdot\tau_{s0}+3j)/60,$$

$$n_1 = 6(\tau_{s0}-\hat{\tau}_{sR}-3j/Re)$$

$$n_2 = 3(\hat{p}_{sR}+mFr-\hat{\tau}_{sI})+Re[Re(6\tau_{s0}^2+8\hat{\tau}_{sR}\tau_{s0}$$
$$+\hat{\tau}_{sR}^2)+6j\tau_{s0}+12j\hat{\tau}_{sR}]/60$$

$$n_3 = 0.4 \quad n_4 = -0.1\cdot Re(2\tau_{s0}+\hat{\tau}_{sR})$$

$$n_5 = 3\hat{p}_{sI}+Re\cdot\hat{\tau}_{sI}(11Re\cdot\tau_{s0}+3Re\cdot\hat{\tau}_{sR}+15j)/60$$

$$n_6 = 3\hat{p}_{sI}+Re\hat{\tau}_{sI}(5Re\cdot\tau_{s0}+Re\cdot\hat{\tau}_{sR}+9j)/60$$

$$n_7 = -Re\cdot\hat{\tau}_{sI}/20 \quad n_8 = (Re\cdot\hat{\tau}_{sI})^2/60,$$

$$n_9 = -3We.$$

As is seen, the determining characteristic parameters of the problem are: film Reynolds number Re, film Weber number We, film Froude number Fr and the unperturbed shear stress τ_{s0}. The stress variation on the interface is taken into account through the given coefficients $\hat{\tau}_{sR}$, $\hat{\tau}_{sI}$, $\hat{p}_{sR}$ and $\hat{p}_{sI}$, which may be determined, for example, according to refs. [10, 11, 14].

Equation (34) has a two-wave structure. This means that at small Reynolds numbers $Re \sim 1$ the wave process is based on the kinematic wave. It is described in the first approximation by the differential equation of the first order:

$$\left(\frac{\partial}{\partial t}+c_0\frac{\partial}{\partial x}\right)H = 0.$$

But at high flow rates ($Re \gg 1$) the dynamical waves dominate which are described by the derivatives of the second order:

$$\left(\frac{\partial}{\partial t}+c_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+c_2\frac{\partial}{\partial x}\right)H = 0.$$

Equations of a two-wave nature and their peculiarities are discussed by Whitham [17].

For a vertical film a two-wave equation has been derived first by Alekseenko *et al.* [16] and Nakoryakov and Alekseenko [18] for an inclined film flow. For the horizontal film entailed by gas flow, the two-wave equation has been obtained by Jurman and McCready [15]. In the limiting and particular cases equation (34) transforms to the known equations [12, 15, 16, 18]. Hence, it is the most general model equation describing the weakly nonlinear long waves in a liquid film moving along an inclined plane under the effect of gravity and gas flow. Note that surface stresses is taken more correctly into account as compared to ref. [15]. In ref. [15] the velocity profile is considered to be

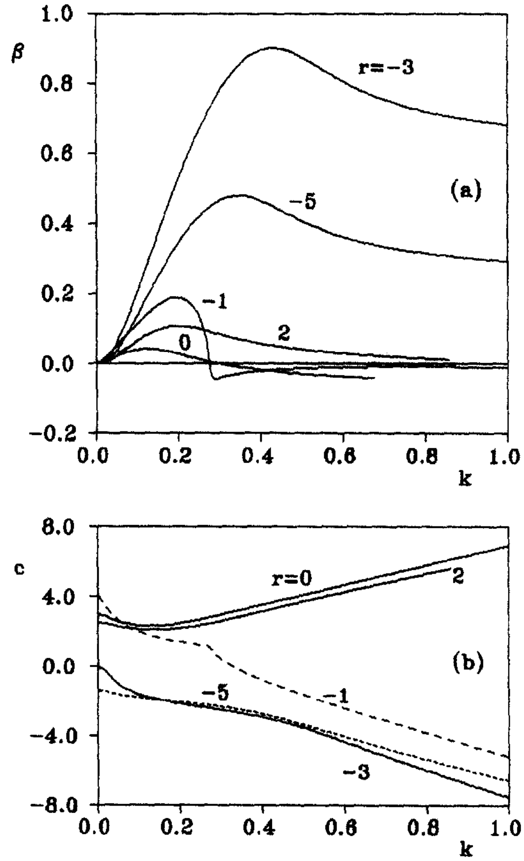

Fig. 1. Dispersion curves for vertical gas–film flow (air–water); $Re = 20$; $Fi^{1/11} = 9.164$.

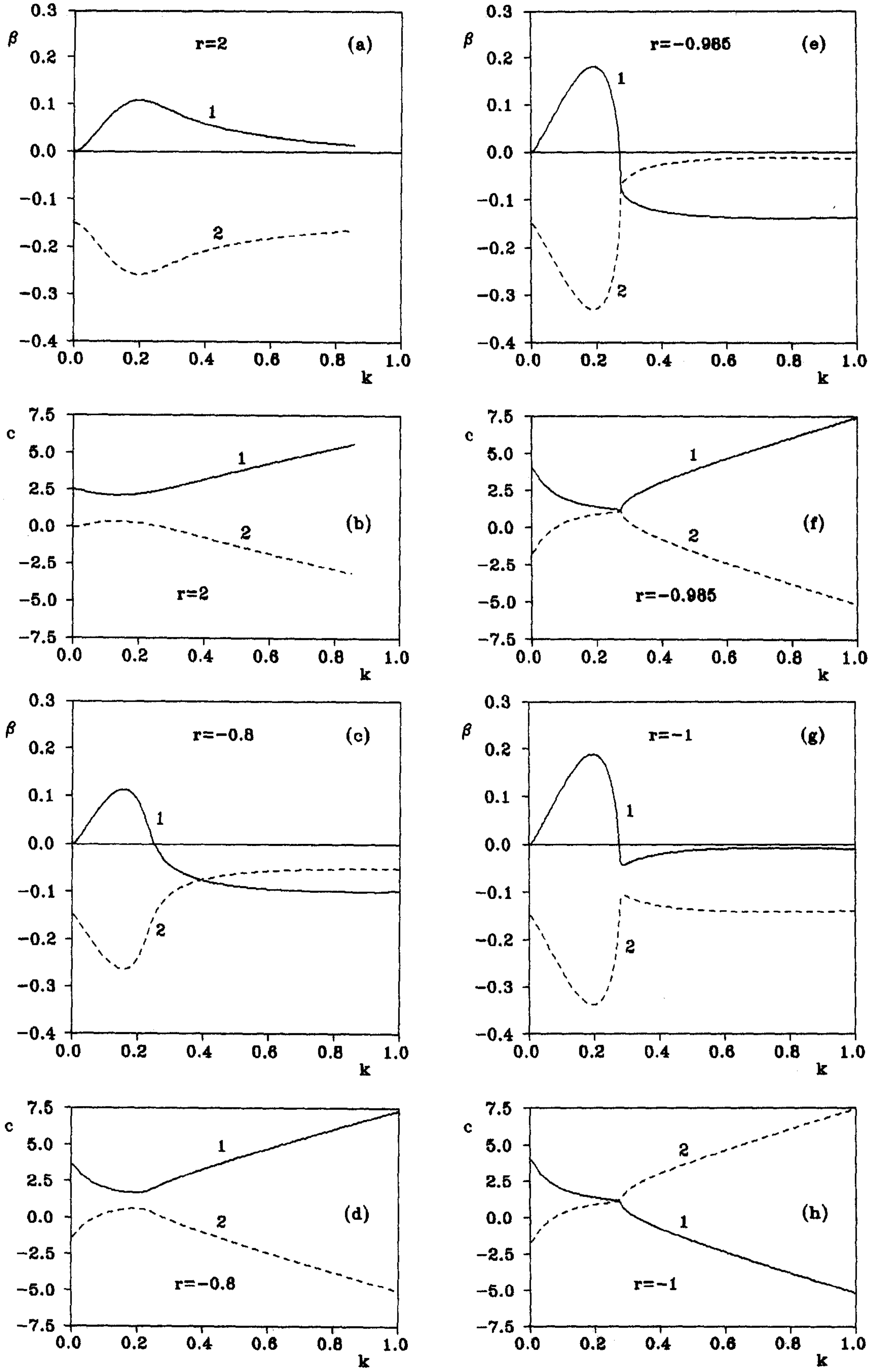

Fig. 2. Full dispersion curves for vertical gas–film flow (air–water); $Re = 20$; $Fi^{1/11} = 9.164$; 1, 2 = 1st and 2nd wave modes.

self-similar. However this is valid only for the limit of small stress on the interface which is of no interest for the problem on a gas–film flow. If we put $\theta = 0$ and neglect perturbations of shape factor κ in equation (23) that is true for the self-similar profile, then the resulting equations in the present study and [15] coincide identically.

5. LINEAR ANALYSIS OF FILM STABILITY

The more precise data on the linear stability of film flow may be obtained from the Orr–Sommerfeld equation [2]. However, we assume it necessary to carry out the linear stability analysis on the basis of the derived equation because only the integral approach is used for calculations of nonlinear waves in a film sheared by gas flow [10].

Representing the film thickness perturbation in the form $H \sim \exp[ik(x-ct)+\beta t]$ and substituting this expression into the linearized wave equation (34), we obtain the dispersion relations:

$$\beta = \frac{3}{Re}\frac{c-c_0+n_0k^2Re/3}{jc_1c_2-2c} \quad (37)$$

$$\beta^2-k^2[c^2-jc_1c_2c+j(c_1+c_2)]+\frac{3\beta}{Re}+We\,k^4 = 0. \quad (38)$$

Here k is the wave number, c is the phase velocity, β is the increment. The neutral curve is defined by the condition: $\beta = 0$. To solve the dispersion relations the pressure and shear stresses at the interface must be given. In this paper, such data are taken from ref. [11], where the quasi-laminar model was used. Abrams and Hanratty [14] have suggested the relaxation model which is more acceptable to describing the interfacial stresses than the quasi-laminar one. However, these theories yield results that do not significantly distinguish themselves in the range of wave numbers used. According to the above models the amplitudes of stress perturbations $\hat{\tau}_{sR}$, $\hat{\tau}_{sI}$, $\hat{p}_{sR}$, $\hat{p}_{sI}$ are the functions of a wave number. Therefore equations (37)–(38) can only be solved numerically.

The dispersion curves for vertical gas-liquid flow are presented in Fig. 1. The influence of the gas stream is taken into account by the parameter $r = Re\cdot\tau_{s0}/(1-0.5\mathrm{Re}\cdot\tau_{s0})$. This parameter is more convenient than τ_{s0} because it defines the direction of film flow as follows: $r > -1.5$—downward flow; $-3 < r < -1.5$—downward flow near the wall and upward flow near the interface; $r < -3$—upward flow. If $r = 0$ the effect of the gas is absent. It follows from the figures that a gas stream always decreases the film stability. The straight line asymptotes for phase velocity c at large k correspond to the capillary waves in shallow water (Fig. 1b).

The peculiarity of the dispersion curves is their unexpected change at some value of r. More detailed calculations explain such behaviour. Every dispersion

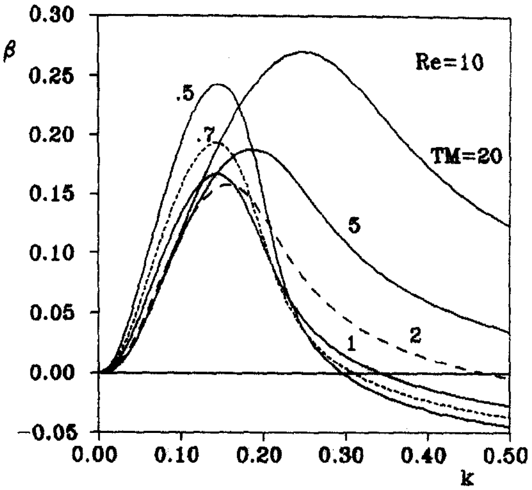

Fig. 3. Growth rate factor β for liquid film sheared by turbulent gas flow along the upper side of a horizontal plane (air–water); $Fi^{1/11} = 9.164$.

curve has two branches which describe different wave modes.

At large or small r, the branches of $(\beta-k)$-dependencies are separated (Fig. 2a, g); in doing so the lower branches correspond to strongly decaying disturbances which are not of interest for the problem of film instability. However, starting with some value of r, the intersecting branches of $(\beta-k)$-dependencies occur (Fig. 2c). With the decreasing parameter r, when critical value is achieved, the branches of the dispersion curves exchange their portions (Fig. 2c–h). As a consequence, the curves accept an unexpectedly complicated shape (Fig. 2g–h). The critical value of r is defined by the conditions under which the branches of (β-k)-dependence have the same inclination angle at the intersection point or the branches of (c-k)-dependence touch one another without intersecting. The described behaviour of curves means that it is necessary to account for both branches of the dispersion curves. A similar problem does not appear for a free falling film when the dispersion curve branches never intersect [4].

A detailed analysis of film instability over a wide range of parameters where the integral theory is valid does not enter into the scope of our paper. We still consider several examples useful for an illustration of the fields of application and for confirmation of the theory.

Figure 3 shows the growth rate factor β vs the wave number k in the case of a horizontal film sheared by the turbulent gas. Here the gas flow parameter TM is defined as $TM = \tau_{s0}(3/gv)^{2/3}/\rho$. The parameter r makes no sense since its value tends to infinity for a horizontal flow. As is seen, a film flow may be stable at a small value of shear stress in contrast to the vertical gas–film flow. The calculations for the neutral curves and the maximum-growing waves are also carried out. The comparison of the obtained results with

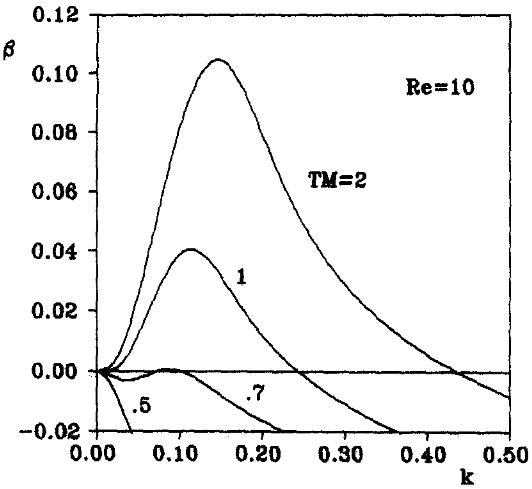

Fig. 4. Growth rate factor β for liquid film sheared by turbulent gas flow along the lower side of a horizontal plane (air–water); $Fi^{1/11} = 9.164$.

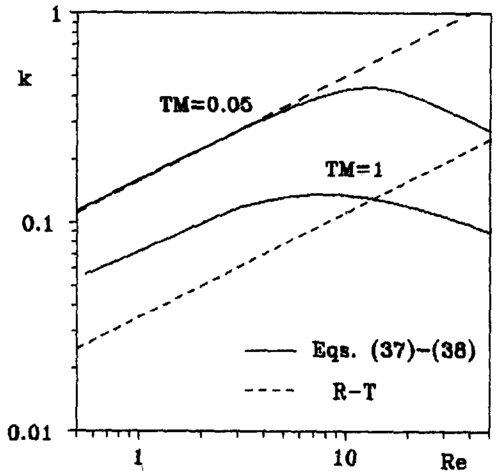

Fig. 5. Neutral curves for liquid film sheared by turbulent gas flow along the lower side of a horizontal plane (air–water); $Fi^{1/11} = 9.164$.

computations [11] on the basis of the Orr–Sommerfeld equation shows good agreement over the range of small wave numbers.

The dispersion curves for a film moving along the lower side of a horizontal plane are presented in Fig. 4. Contrary to a flow on the upper side (Fig. 3), here the film flow is always unstable and the effect of parameter TM on the instability is nonmonotonic. It is interesting to compare the viscous film instability in the presence of a gas stream with the Rayleigh–Taylor one (R–T). Such a comparison is made in Fig. 5 where neutral curves are shown. It follows from this figure that R–T instability occurs in the range of small values of Re and TM ($TM = 0.05$, $Re < 5$). If the shear stress is sufficiently large ($TM = 1$), the gas stream has a destabilizing effect on the film instability at $Re > 15$, and opposite action at $Re < 15$.

The obtained results demonstrate the possibility of applying the two-wave equation to modelling linear and nonlinear waves on a liquid film moving under the effect of gravity and turbulent gas flow over wide range of conditions.

REFERENCES

1. G. F. Hewitt and M. S. Hall-Taylor, *Annular Two-Phase Flow*. Pergamon Press, Oxford (1970).
2. T. J. Hanratty, Interfacial instabilities caused by air flow over a thin liquid layer. In *Waves on Fluid Interfaces*, pp. 221–259. Academic Press, London (1983).
3. T. J. Hanratty, Separated flow modelling and interfacial transport phenomena, *Appl. Sci. Res.* **48,** 353–390 (1991).
4. S. V. Alekseenko, V. E. Nakoryakov and B. G. Pokusaev, *Wave Flow of Liquid Films*. Begell, New York (1994).
5. V. E. Nakoryakov, B. G. Pokusaev and I. R. Shreiber, *Wave Propagation in Gas–Liquid Media*. Begell, New York (1992).
6. J. M. Miles, On the generation of surface waves by shear flows, *J. Fluid Mech.* **3,** 185 (1957).
7. T. B. Benjamin, Shearing flow over a wavy boundary, *J. Fluid Mech.* **6,** 161–205 (1959).
8. L. S. Cohen and T. J. Hanratty, Generation of waves in the concurrent flow of air and a liquid, *AIChE J.* **11,** 138–144 (1965).
9. A. D. D. Craik, Wind-generated waves in liquid films, *J. Fluid Mech.* **26,** 369–392 (1969).
10. E. A. Demekhin, G. Yu. Tokarev and V. Ya. Shkadov, Instability and nonlinear waves in a vertical liquid film flowing with countercurrent turbulent gas stream *Teor. Osnovy Khim. Tekhn.* **23,** 64–70 (1989) (in Russian).
11. E. A. Demekhin and O. L. Potapov, The mathematical simulation of the wavy thin layers with exterior active forces hydrodynamic. Preprint N215, IT SO AN SSSR, Novosibirsk (1990) (in Russian).
12. L. P. Kholpanov and V. Ya. Shkadov, *Hydrodynamic and Heat and Mass Transfer with Free Surface*. Nauka, Moscow (1990) (in Russian).
13. S. V. Alekseenko, V. E. Nakoryakov and B. G. Pokusaev, Wave formation on a vertically falling liquid film, *AIChE J.* **32,** 1446–1460 (1985).
14. J. Abrams and T. J. Hanratty, Relaxation effects observed for turbulent flow over a wavy surface, *J. Fluid Mech.* **151,** 443–455 (1985).
15. L. A. Jurman and M. J. McCready, Study of waves on thin liquid films sheared by turbulent gas flows, *Phys. Fluids A* **1,** 522–536 (1989).
16. S. V. Alekseenko, V. E. Nakoryakov and B. G. Pokusaev, Wave formation in liquid film flow on a vertical wall, *J. Appl. Mech. Technol. Phys.* **20,** 719–727 (1979).
17. G. B. Whitham, *Linear and Nonlinear Waves*. Wiley, New York (1974).
18. V. E. Nakoryakov and S. V. Alekseenko, Waves on a film flowing down an incline, *Fluid Mech. Sov. Res.* **10,** 18–33 (1981).